AMS210.01.
Homework 6

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In this homework we will consider the following scalar products

- In $\mathbb{R}^n$: $\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$;
- In $M_{m,n}$: $\langle A, B \rangle = \mathrm{tr}(AB^\top)$;
- In $C[a, b]$: $\langle f, g \rangle = \int_a^b f(t) g(t)\, dt$.

In this homework there are a lot of extra-credit problems of different levels of difficulty. You should understand perfectly how to solve standard problems, and proceed to extra-credit problems only after that! The problems from exam and quizes will include only standard problems.

1. Compute $\langle 3u - 5v, 2u + v \rangle$ if $\langle u, u \rangle = 5$, $\langle u, v \rangle = 1$ and $\langle v, v \rangle = 2$.

2. Compute the following scalar products:

   (a) $\langle (2, 1, -3, 1), (0, 4, 2, 2) \rangle$ in $\mathbb{R}^4$ with standard scalar product.

   (b) $\langle 2t + 1, t^2 \rangle$ in the space $C[0, 1]$.

   (c) $\langle 2t + 1, t^2 \rangle$ in the space $C[-1, 1]$.

3. Find norms and normalizations of the following vectors:

   (a) $(4, 2, 2)$ in $\mathbb{R}^3$ with standard scalar product.

   (b) t^2 in $C[0, 1]$.

   (c) t^2 in $C[-1, 1]$.

4. Find the cosines of the angles between the following vectors:

   (a) $(0, 2)$ and $(3, -3)$.

   (b) t^2 and $t^2 + 1$ in $C[0, 1]$.

   (c) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$ in $M_{2,3}$.

5. Find all constants a such that

   (a) vectors $(a, 2)$ and $(a, -8)$ are orthogonal in $\mathbb{R}^2$.

(b) vectors t^2 and $t^2 + a$ are orthogonal in $C[0, 1]$.

6. Determine the distances between the following points:

   (a) $(1, 2, -4)$ and $(0, 2, 5)$.

   (b) $2t - 1$ and $3t + 1$ in $C[0, 1]$.

7. Find all values of a such that $\|(1, a, -3, 2)\| = 5$.

8. Prove that the following pairs of vectors are orthogonal:

   (a) $(1, 4, -2)$ and $(2, 1, 3)$ in $\mathbb{R}^3$.

   (b) $\cos t$ and $\sin 2t$ in $C[-\pi, \pi]$.

   (c) $3t^2 - 1$ and $5t^3 - 3t$ in $C[-1, 1]$.

9. Find the vector $v = (a, b, c)$ which is orthogonal to the both vectors $v_1 = (1, 2, 1)$ and $v_2 = (1, -1, 1)$.

10. Find the bases of the orthogonal compliment $S^{\perp}$ of the following sets of vectors S:

    (a) $S = \{(1, 4, 5, 2)\}$.

    (b) $S = \{(1, -2, 2, 4, 1), (2, -2, -1, 0, 4)\}$.

11. Find the coordinates of the vector v in basis consisting of u_i's, if it is known that the basis is orthogonal. Use the method, specific for orthogonal bases, otherwise you will not get a credit!

    (a) $v = (2, 3)$, basis: $u_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $u_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

    (b) $v = (1, 2, 3)$, basis: $u_1 = (1, -1, 2)$, $u_2 = (2, 2, 0)$, and $u_3 = (-1, 1, 1)$.

12. Compute projections of the vector v to the vector w:

    (a) $v = (1, 2, 3)$, $w = (2, 2, 2)$.

    (b) t to $t^2 + 1$ in $C[0, 1]$.

13. (a) Compute projection of the vector $(-1, 0, 1)$ to the plane with the basis $\{(0, 1, 0), (\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}})\}$.

    (b) Find the distance between $(-1, 0, 1)$ and the plane from the previous part.

14. (a) Compute projection of the following vector $(4, -1, -3, 4)$ to the subspace with the following basis $\{(1, 1, 1, 1), (1, 2, 2, -1), (1, 0, 0, 3)\}$.

    (b) Find the distance between $(4, -1, -3, 4)$ and the subspace from the previous part.

15. Apply the Gram-Schmidt orthogonalization process to the following sets of vectors:

    (a) $(1, -1, 0, 1)$, $(2, 0, 0, 1)$, $(0, 0, 1, 0)$.

    (b) $(1, 1, -1, 0)$, $(0, 2, 0, 1)$, $(-1, 0, 0, 1)$.

16. **[Extra credit]** Determine which of the following functions are bilinear:

   (a) $f(A, B) = \text{tr}(AB)$, $A, B \in M_{n,n}$.

   (b) $f(A, B) = \text{tr}(AB - BA)$.

   (c) $f(A, B) = \det AB$.

   (d) $f(A, B) = \text{tr}(A + B)$.

   (e) $f(A, B) = \text{tr}(AB^\top)$.

   (f) $f(A, B) = (i, j)$-th element of AB.

   (g) $f(u, v) = \int_a^b u(t)v(t)\, dt$, $u, v \in C[a, b]$.

   (h) $f(u, v) = \int_a^b (u(t) + v(t))^2\, dt$.

   (i) $f(u, v) = (uv)'(a)$, a is a fixed number.

17. **[Extra credit]** Using the scalar products prove the following fact:

   (a) The sum of the squares of the diagonals of the parallelogram is equal to the sum of the squares of its sides.

   (b) If a, b, and c are sides of the triangle, then $c^2 = a^2 + b^2 - 2ab\cos\alpha$, where α is the angle between a and b.

18. **[Extra credit]**

   (a) Find the length of the diagonal of the n-dimensional cube with the side a. (Hint: use Pythagoras theorem!)

   (b) Find the radius R of the sphere, circumscribed around the n-dimensional cube with the side a. Find when R is less then a.

19. **[Extra credit]** Find the length of the orthogonal projection of the side of the n-dimensional cube to its diagonal.

20. **[Extra credit]** Find the angle between the vector x and the subspace L, if $x = (2, 2, 1, 1)$ and L is the plane based on the following two vectors: $(3, 4, -4, -1)$ and $(0, 1, -1, 2)$. (Hint: the angle between the vector and the subspace is equal to the angle between the vector and its projection to the given subspace)

21. **[Extra credit]**

   (a) Apply the Gram-Schmidt orthogonalization process to the polynomials 1, t, t^2, and t^3 in the space $C[0, 1]$.

   (b) Find the projection of t^5 onto the subspace, spanned by $1, t, t^2, t^3$.

22. **[Extra credit]** The function $v \mapsto \|v\|$ on the vector space is called **norm** if it satisfies the following properties:

   (i) $\|v\| \geq 0$; if $\|v\| = 0$, then $v = \mathbf{0}$.

(ii) $\|kv\| = |k|\|v\|$.

(iii) $\|u + v\| \leq \|u\| + \|v\|$.

We can define norms in different ways. Let's define the following 3 functions in $\mathbb{R}^n$:

$$\|(a_1, a_2, \ldots, a_n)\|_\infty = \max_i(|a_i|);$$
$$\|(a_1, a_2, \ldots, a_n)\|_1 = |a_1| + |a_2| + \cdots + |a_n|;$$
$$\|(a_1, a_2, \ldots, a_n)\|_2 = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

(a) Prove that all these functions are norms in $\mathbb{R}^n$.

(b) Describe the unit circles on the plane in these norms, i.e. the sets of points $u = (x, y)$ such that $\|u\|_\infty = 1$, $\|u\|_1 = 1$, and $\|u\|_2 = 1$.

23. **[Extra credit]** Prove that the following vectors form an orthogonal system in the space $C[-\pi, \pi]$:

$$1, \sin t, \sin 2t, \sin 3t, \ldots, \cos t, \cos 2t, \cos 3t, \ldots.$$

24. **[Extra credit]** Find the projection of the function e^t onto the space of polynomials $P_2(t)$, if the scalar product is defined as $\langle u, v\rangle = \int_{-1}^{1} u(t)v(t)\,dt$

25. **[Extra credit]** Suggest an algorithm of finding a distance from the point to the plane, for example try to solve the following problem: find the distance from the point $a = (4, 1, -4, -5)$ to the plane $P = (3, -2, 1, 5) + \langle(2, 3, -2, -2), (4, 1, 3, 2)\rangle$ (This is the plane which goes through the point $(3, -2, 1, 5)$ and is generated by vectors $(2, 3, -2, -2)$ and $(4, 1, 3, 2)$).

26. **[Extra credit]** Suggest an algorithm of finding a line which goes through the given point and is orthogonal to the given plane, for example, find the line (i.e., vector on it) which goes through the point $a = (5, -4, 4, 0)$ and is orthogonal to the plane $P = (2, -1, 2, 3) + \langle(1, 1, 1, 2), (2, 2, 1, 1)\rangle$.